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Physical Propositions and Quantum Languages

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Abstract The word *proposition* is used in physics with different meanings, which must be distinguished to avoid interpretational problems. We construct two languages $\mathcal{L}^*(x)$ and $\mathcal{L}(x)$ with classical set-theoretical semantics which allow us to illustrate those meanings and to show that the non-Boolean lattice of propositions of quantum logic (QL) can be obtained by selecting a subset of *p-testable* propositions within the Boolean lattice of all propositions associated with sentences of $\mathcal{L}(x)$. Yet, the aforesaid semantics is incompatible with the standard interpretation of quantum mechanics (QM) because of known no-go theorems. But if one accepts our criticism of these theorems and the ensuing SR (semantic realism) interpretation of QM, the incompatibility disappears, and the classical and quantum notions of truth can coexist, since they refer to different metalinguistic concepts (*truth* and *verifiability according to QM*, respectively). Moreover one can construct a quantum language $\mathcal{L}_{TQ}(x)$ whose Lindenbaum–Tarski algebra is isomorphic to QL, the sentences of which state (testable) properties of individual samples of physical systems, while standard QL does not bear this interpretation.

Keywords Quantum mechanics · Proposition · Classical truth · Quantum truth · Verifiability

1 Introduction

The word *proposition* has been used in physics with some different meanings. Jauch [20] intended it simply as a synonym of *yes-no experiment*, Piron [23] denoted by it an equivalence class of *questions*, etc., following a tradition started by Birkhoff and von Neumann [4] with their *experimental propositions*. On the other hand, the same term is also used in order to denote the (closed) set of states associated with an experimental proposition, often called *physical proposition* (see, e.g., [7], Introduction to Part I). The latter use is commonly

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preferred by those logicians concerned with quantum logic (QL) who identify states with *possible worlds* (ibid., Chap. 8). For, an experimental proposition can be considered as a sentence of a physical language, and the set of states associated with it as its proposition in a standard logical sense. However, the term *proposition* is also used to denote an element of the Lindenbaum–Tarski algebra of the aforesaid physical language (see, e.g., [25], Chap. 5; the links between these meanings are rather obvious).

Let us adopt from now on the standard logical meaning of the term proposition, accepting to identify physical states with possible worlds (which may be questioned from several viewpoints; we, however, do not want to discuss this topic in the present paper). Then, a serious problem occurs when dealing with quantum mechanics (QM), hence with QL. Indeed, every Birkhoff and von Neumann's experimental proposition can be experimentally confirmed or refuted (see also [19], Chap. 8), so that it can be interpreted as a sentence α of an observative language, stating a physical property that can be tested on one or more individual samples of a given physical system (*physical objects*). In classical mechanics (CM) a truth value is defined for every (atomic or molecular) sentence α, and the physical proposition p_α of α (meant as a set of states in which α is true) is introduced basing on this definition. On the contrary, it can occur in QM that no truth value can be defined for a sentence α because of *nonobjectivity of properties* (equivalently, the distinction between *actual* and *potential* properties), which is a well known and debated feature of this theory (see, e.g., [5], Chap. II; [22]). Indeed, nonobjectivity prohibits one to associate a physical property E with a set of physical objects possessing E, which is a basic step if one wants to construct a classical set-theoretical semantics. Hence, a physical proposition is directly associated, in QM, with α, whose truth value is defined via the proposition itself. This gives rise to a number of difficulties, since the notion of truth introduced in this way has several odd features. For instance, if a sentence is not true in a possible world (state), one cannot assert that it is false in that world, and the join of two sentences may be true even if none of the sentences is true. More important, this notion of truth clashes with the fact that every (elementary) experimental proposition can be checked on a physical object, yielding one of two values (0 or 1) that can be intuitively interpreted as *true* and *false*. Thus, the identification of sentences with their propositions may produce serious troubles (the "metaphysical disaster" pointed out, though in a somewhat different way, by Foulis and Randall [24]). According to Dalla Chiara et al. ([7], Chap. 1) this problem stimulated the investigation about more and more general quantum structures. In our opinion, however, the attempt at solving it in this way is questionable. Indeed, the problem is originated by some specific features of the standard interpretation of the mathematical formalism of QM (to be precise, the aforesaid nonobjectivity of properties) and not by the formalism itself, so that it cannot be solved by simply generalizing the mathematical apparatus without removing those peculiarities of the interpretation that create it (see also [6]).

According to a widespread belief, the impossibility of solving the above problem by firstly endowing the language of QM with a classical set-theoretical semantics and then introducing the set of propositions is witnessed by the fact that this set has a structure of orthomodular nondistributive lattice, while a classical semantics would lead to a Boolean lattice of propositions.

We aim to show in this paper that the above belief is ill-founded. To be precise, we want to show that one can construct a simple language $\mathcal{L}(x)$ endowed with a classical set-theoretical semantics, associate it with a poset of *physical propositions* (that generally is not a lattice), and then introduce a definition of *testability* on $\mathcal{L}(x)$ which selects a subposet of *testable* (actually, *p-testable*, see Sect. 4) *physical propositions*. Our procedure is very intuitive, and applies to every theory, as CM and QM, in which physical objects and properties can be defined. Under reasonable physical assumptions the poset of all testable physical propositions turns out to be a Boolean lattice in CM, while it is an orthomodular nondistributive lattice in QM that can be identified with a (standard, sharp) QL. It follows, in particular, that nondistributivity cannot be considered an evidence that a classical notion of truth cannot be introduced in QM.

Our result does not prove, of course, that providing a classical semantics for the observative language of QM is actually possible. Indeed, nonobjectivity of properties would still forbid it. However, should one accept the criticism to nonobjectivity provided by ourselves in some previous paper, and the *Semantic Realism* (SR) interpretation of QM following from it (see [9–15]),[1] the language $\mathcal{L}(x)$ introduced in this paper appears as a sublanguage of the broader observative language of QM, and the classical set-theoretical semantics defined on it can be seen as a restriction of the broader classical set-theoretical semantics that can be defined on the observative language. If this viewpoint is accepted, the distinction between physical propositions and testable physical propositions can be considered something more than an abstract scheme for showing how non-Boolean algebras can be recovered within a Boolean framework. Indeed, physical propositions are then associated in a standard way with (universally) quantified sentences of $\mathcal{L}(x)$ that have classical truth values, which avoids the "metaphysical disaster" mentioned above, and testable physical propositions are physical propositions associated with quantified sentences for which *truth criteria* are given that allow one to determine empirically their truth values.[2]

The lattice operations on the lattice of all testable physical propositions, however, only partially correspond to logical operations of $\mathcal{L}(x)$ in QM. We show that $\mathcal{L}(x)$ can be enriched by introducing new *quantum connectives*, so that a language $\mathcal{L}_{TQ}(x)$ of testable sentences can be extracted from $\mathcal{L}(x)$ whose Lindenbaum–Tarski algebra is isomorphic to the orthomodular lattice of all testable physical propositions of $\mathcal{L}(x)$. Thus, we introduce a clear distinction between classical and quantum connectives, and show that a verificationist notion of *quantum truth* can be defined on $\mathcal{L}_{TQ}(x)$ which coexists with the classical definition of truth, rather than being alternative to it. This is a noticeable achievement, which avoids postulating that different incompatible notions of truth are implicitly introduced by our physical reasonings.

Some of the results resumed above have already been expounded in some previous papers [16, 17], though in a somewhat different form. Here we generalize our previous treatments

[1] We remind that our criticism is based on an epistemological perspective according to which the *theoretical laws* of any physical theory are considered as mathematical schemes from which *empirical laws* can be deduced. The latter laws are assumed to be valid in all those physical situations in which they can be experimentally checked, while no assumption of validity can be done in physical situations in which some general principle prohibits one to check them (this position is consistent, in particular, with the operational and antimetaphysical attitude of standard QM). In CM our perspective does not introduce any substantial change, since there is no physical situation in which an empirical law cannot, in principle, be tested. On the contrary, if boundary, or initial, conditions are given in QM which attribute noncompatible properties to the physical system (more precisely, to a sample of it), a physical situation is hypothesized that cannot be empirically accessible, hence no assumption of validity can be done for the empirical laws deduced from the general formalism of QM in this situation. Strangely enough, this new perspective is sufficient to invalidate the proof of some important no-go theorems, as Bell's [1] and Bell–Kochen–Specker's [2, 18]. Nonobjectivity of properties then appears in this context as an interpretative choice, not a logical consequence of the theory, and alternative interpretations become possible. Among these, our SR interpretation restores objectivity of properties without requiring any change in the mathematical apparatus and in the minimal (statistical) interpretation of QM.

[2] From a logical viewpoint our treatment exhibits the deep reasons of the "disaster". Indeed, *experimental* propositions are interpreted as open sentences of a first order predicate language, while *physical* propositions are associated with quantified sentences of the same language.

by considering effects in place of properties, which leads us to preliminarily construct a broader language $\mathcal{L}^*(x)$ in which $\mathcal{L}(x)$ is embedded. An interesting consequence of this broader perspective is a weakening of the notion of testability, which illustrates from our present viewpoint a possible advantage of unsharp QM with respect to standard QM. We also provide a simple new way for defining physical propositions by introducing universal quantifiers on the sentences of the language $\mathcal{L}^*(x)$, which also helps in better understanding the notion of quantum truth and its difference from classical truth. For the sake of brevity, however, our presentation is very schematic and essential.

It remains to observe that a more general treatment of the topics discussed in this paper could be done by adopting the formalization of an observative sublanguage of QM introduced by ourselves many years ago [8]. In this case, two classes of predicates would occur, one denoting effects (hence properties), one denoting states, so that states would not be identified with possible worlds and physical propositions would be distinguished from propositions in a standard logical sense. This treatment would be more general and formally complete, at the expense, however, of simplicity and understandability, so that we do not undertake this task here.

2 The Language of Effects $\mathcal{L}^*(x)$

We call $\mathcal{L}^*(x)$ the formal language constructed by means of the following symbols and rules.

Alphabet.
An individual variable x.
Monadic predicates $E, F, \ldots$.
Logical connectives $\neg, \wedge, \vee$.
Auxiliary signs (,).

Syntax.
Standard classical formation rules for well formed formulas (briefly, *wffs*).

We introduce a *set-theoretical semantics* on $\mathcal{L}^*(x)$ by means of the following metalinguistic symbols, sets and rules.

$\mathcal{E}^*$: the set of all predicates.

$\Phi^*(x)$: the set of all wffs of $\mathcal{L}^*(x)$.

$\mathcal{E}^*(x)$: the set $\{E(x) \mid E \in \mathcal{E}^*\}$ of all *elementary* wffs of $\mathcal{L}^*(x)$.

A set $\mathcal{S}$ of *states*.

For every $S \in \mathcal{S}$, a universe $\mathcal{U}_S$ of *physical objects*.

A set $\mathcal{R}$ of mappings (*interpretations*) such that, for every $\rho \in \mathcal{R}$, $\rho : (x, S) \in \{x\} \times \mathcal{S} \to \rho_S(x) \in \mathcal{U}_S$.

For every $S \in \mathcal{S}$ and $E \in \mathcal{E}^*$, an *extension* $\text{ext}_S(E) \subseteq \mathcal{U}_S$.

For every $\rho \in \mathcal{R}$ and $S \in \mathcal{S}$, a *classical assignment function* $\sigma_S^\rho : \Phi^*(x) \to \{t, f\}$ (where t stands for *true* and f for *false*), defined according to standard (recursive) truth rules in Tarskian semantics (to be precise, for every elementary wff $E(x) \in \mathcal{E}^*(x)$, $\sigma_S^\rho(E(x)) = t$ iff $\rho_S(x) \in \text{ext}_S(E)$, for every pair $\alpha(x)$, $\beta(x)$ of wffs of $\Phi^*(x)$, $\sigma_S^\rho(\alpha(x) \wedge \beta(x)) = t$ iff $\sigma_S^\rho(\alpha(x)) = t = \sigma_S^\rho(\beta(x))$, etc.).

The *intended physical interpretation* of $\mathcal{L}^*(x)$ can then be summarized as follows.

Reference to a physical system Σ is understood.

A predicate of $\mathcal{L}^*(x)$ denotes an *effect*, which is operationally interpreted as an equivalence class of (dichotomic) *registering devices*, each of which, when activated by an individual sample of Σ, performs a *registration* that may yield value 0 or 1 (see, e.g., [21], Chap. II; [14, 15]). We assume in the following that every registering device belongs to an effect.

A state is operationally interpreted as an equivalence class of *preparing devices*, each of which, when activated, performs a *preparation* of an individual sample of Σ (ibid.).

A physical object is operationally interpreted as an individual sample of Σ, which can be identified with a preparation (ibid.).

The equation $\sigma_S^\rho(E(x)) = t$ (or f) is interpreted as meaning that, if a registering device belonging to E is activated by the physical object $\rho_S(x)$, the result of the registration is 1 (or 0). The interpretation of $\sigma_S^\rho(\alpha(x)) = t$ (or f), with $\alpha(x) \in \Phi^*(x)$, follows in an obvious way, bearing in mind the above truth rules for the connectives $\neg, \wedge, \vee$.

Let us now introduce some further definitions and notions.

(i) We define a *logical preorder* $<$ and a *logical equivalence* $\equiv$ on $\Phi^*(x)$ in a standard way, as follows.

Let $\alpha(x), \beta(x) \in \Phi^*(x)$. Then,

$$\alpha(x) < \beta(x) \quad \text{iff} \quad \text{for every } \rho \in \mathcal{R} \text{ and } S \in \mathcal{S},\ \sigma_S^\rho(\alpha(x)) = t \text{ implies } \sigma_S^\rho(\beta(x)) = t,$$

$$\alpha(x) \equiv \beta(x) \quad \text{iff} \quad \alpha(x) < \beta(x) \text{ and } \beta(x) < \alpha(x).$$

We note that the quotient set $\Phi^*(x)/\equiv$ is partially ordered by the order (still denoted by $<$) canonically induced on it by the preorder $<$. It easy to prove that the poset $(\Phi^*(x)/\equiv, <)$ is a Boolean lattice.

(ii) Let $\alpha(x) \in \Phi^*(x)$. We call *physical sentence associated with* $\alpha(x)$ the (universally) quantified sentence $(\forall x)\alpha(x)$, and denote by Ψ^* the set of all physical sentences associated with wffs of $\mathcal{L}^*(x)$ (hence $\Psi^* = \{(\forall x)\alpha(x) \mid \alpha(x) \in \Phi^*(x)\}$). Then, for every $S \in \mathcal{S}$, we introduce a *classical assignment function* $\sigma_S : \Psi^* \to \{t, f\}$ by setting, for every physical sentence $(\forall x)\alpha(x) \in \Psi^*$,

$$\sigma_S((\forall x)\alpha(x)) = t \quad \text{iff} \quad \text{for every } \rho \in \mathcal{R},\ \sigma_S^\rho(\alpha(x)) = t.$$

The logical preorder and equivalence defined on $\Phi^*(x)$ can be extended to Ψ^* in a standard way, as follows.

Let $(\forall x)\alpha(x), (\forall x)\beta(x) \in \Psi^*$. Then,

$$(\forall x)\alpha(x) < (\forall x)\beta(x) \quad \text{iff} \quad \text{for every } S \in \mathcal{S},\ \sigma_S((\forall x)\alpha(x)) = t \text{ implies } \sigma_S((\forall x)\beta(x)) = t,$$

$$(\forall x)\alpha(x) \equiv (\forall x)\beta(x) \quad \text{iff} \quad (\forall x)\alpha(x) < (\forall x)\beta(x) \text{ and } (\forall x)\beta(x) < (\forall x)\alpha(x).$$

The quotient set $\Psi^*/\equiv$ is partially ordered by the order (still denoted by $<$) canonically induced on it by the preorder $<$, but the poset $(\Psi^*/\equiv, <)$ is not bound to be a lattice.

(iii) We use the definitions in (ii) to introduce a notion of *true with certainty* on $\Phi^*(x)$. For every $\alpha(x) \in \Phi^*(x)$ and $S \in \mathcal{S}$, we put

$\alpha(x)$ is *certainly true* in S iff $\sigma_S((\forall x)\alpha(x)) = t$

(equivalently, the physical sentence $(\forall x)\alpha(x))$ associated with $\alpha(x)$ is *true*).

A wff $\alpha(x) \in \Phi^*(x)$ can be certainly true in the state S or not. It must be stressed that in the latter case we do not say that $\alpha(x)$ is *certainly false* in S: this term will be introduced indeed at a later stage, with a different meaning. We also note explicitly that the new truth value is attributed or not to a wff of $\Phi^*(x)$ independently of a specific interpretation ρ.

The notion of true with certainty allows one to introduce a *physical preorder* $\prec$ and a *physical equivalence* $\approx$ on $\Phi^*(x)$, as follows.

Let $\alpha(x), \beta(x) \in \Phi^*(x)$. Then,

$$\alpha(x) \prec \beta(x) \quad \text{iff} \quad \text{for every } S \in \mathcal{S},\ \alpha(x) \text{ certainly true in } S \text{ implies}$$

$$\beta(x) \text{ certainly true in } S \text{ (equivalently, } (\forall x)\alpha(x)) < (\forall x)\beta(x)).$$

$$\alpha(x) \approx \beta(x) \quad \text{iff} \quad \alpha(x) \prec \beta(x) \text{ and } \beta(x) \prec \alpha(x) \text{ (equivalently, } (\forall x)\alpha(x) \equiv (\forall x)\beta(x)).$$

It is apparent that the logical preorder $<$ and the logical equivalence $\equiv$ on $\Phi^*(x)$ imply the physical preorder $\prec$ and the physical equivalence $\approx$, respectively, while the converse implications generally do not hold. Moreover, one can introduce the quotient set $\Phi^*(x)/\approx$, partially ordered by the order (still denoted by $\prec$) canonically induced on it by the preorder $\prec$ defined on $\Phi^*(x)$. Then, the posets $(\Phi^*(x)/\approx, \prec)$ and $(\Psi^*/\equiv, <)$ are obviously order-isomorphic.

(iv) We want to introduce a concept of *testability* on $\Phi^*(x)$. To this end, let us consider an elementary wff $E(x) \in \Phi^*(x)$ and observe that it is testable in the sense that its truth value for a given interpretation ρ and state S can be empirically checked by using one of the registering devices in the class denoted by E in order to perform a registration on $\rho_S(x)$. Let us consider now a molecular wff $\alpha(x)$ of $\Phi^*(x)$ and agree that it is testable iff a registering device exists that allows us to check its truth value. Since we have assumed that every registering device belongs to an effect, we conclude that $\alpha(x)$ is testable iff it is logically equivalent to an elementary wff of $\Phi^*(x)$. Thus, we introduce the subset $\Phi_T^*(x)$ of all testable wffs of $\Phi^*(x)$, defined as follows.

$$\Phi_T^*(x) = \{\alpha(x) \in \Phi^*(x) \mid \exists E_\alpha \in \mathcal{E}^* : \alpha(x) \equiv E_\alpha(x)\}.$$

Of course, the binary relations $<$, $\equiv$, $\prec$ and $\approx$ introduced on $\Phi^*(x)$ can be restricted to $\Phi_T^*(x)$, and we still denote these restrictions by the symbols $<$, $\equiv$, $\prec$ and $\approx$, respectively, in the following.

(v) The notion of testability can be extended to the physical sentences associated with wffs of $\Phi^*(x)$ by setting, for every $\alpha(x) \in \Phi^*(x)$,

$$(\forall x)\alpha(x) \text{ is testable} \quad \text{iff} \quad \alpha(x) \text{ is testable (equivalently, } \alpha(x) \in \Phi_T^*(x)).$$

We denote the set of all testable physical sentences by Ψ_T^* (hence, $\Psi_T^* = \{(\forall x)\alpha(x) \mid \alpha(x) \in \Phi_T^*(x)\}$), and still denote the restrictions to Ψ_T^* of the binary relations $<$ and $\equiv$ defined on Ψ^* by $<$ and $\equiv$, respectively. It is then easy to show that the posets $(\Phi_T^*(x)/\approx, \prec)$ and $(\Psi_T^*/\equiv, <)$ are order-isomorphic.

3 Physical Propositions

Let $\alpha(x) \in \Phi^*(x)$. We put

$$p_\alpha^f = \{S \in \mathcal{S} \mid \alpha(x) \text{ is certainly true in } S\},$$

and say that p_α^f is the *physical proposition associated with* $\alpha(x)$ (or, briefly, the *physical proposition of* $\alpha(x)$). It is then easy to see that p_α^f is the proposition associated with

$(\forall x)\alpha(x)$ according to the standard rules of a Kripkean semantics in which states play the role of possible worlds. More formally,

$$p_\alpha^f = \{S \in \mathcal{S} \mid \sigma_S((\forall x)\alpha(x)) = t\} = \{S \in \mathcal{S} \mid \text{for every } \rho \in \mathcal{R},\ \sigma_S^\rho(\alpha(x)) = t\}.$$

We denote by $\mathcal{P}^{*f}$ the set of all physical propositions of wffs of $\Phi^*(x)$,

$$\mathcal{P}^{*f} = \{p_\alpha^f \mid \alpha(x) \in \Phi^*(x)\}.$$

The definitions of certainly true in S, physical order $\prec$ and physical equivalence $\approx$ can be restated by using the notion of physical proposition. Indeed, for every $\alpha(x), \beta(x) \in \Phi^*(x)$,

$$\alpha(x) \text{ is certainly true in } S \quad \text{iff} \quad S \in p_\alpha^f,$$
$$\alpha(x) \prec \beta(x) \quad \text{iff} \quad p_\alpha^f \subseteq p_\beta^f,$$
$$\alpha(x) \approx \beta(x) \quad \text{iff} \quad p_\alpha^f = p_\beta^f.$$

The above results imply that the posets $(\Phi^*(x)/\approx, \prec)$ (or $(\Psi^*/\equiv, <)$) and $(\mathcal{P}^{*f}, \subseteq)$ are order-isomorphic.[3] However, the set-theoretical operations on $\mathcal{P}^{*f}$ do not generally correspond to logical operations on $\Phi^*(x)$. Indeed, for every $\alpha(x), \beta(x), \gamma(x) \in \Phi^*(x)$, one gets

$$\alpha(x) \equiv \neg\beta(x) \quad \text{implies} \quad p_\alpha^f \subseteq \mathcal{S} \setminus p_\beta^f,$$
$$\alpha(x) \equiv \beta(x) \wedge \gamma(x) \quad \text{implies} \quad p_\alpha^f = p_\beta^f \cap p_\gamma^f,$$
$$\alpha(x) \equiv \beta(x) \vee \gamma(x) \quad \text{implies} \quad p_\alpha^f \supseteq p_\beta^f \cup p_\gamma^f$$

(see also [17]).

Let us consider now the subset $\Phi_T^*(x)$ of all testable wffs of $\Phi^*(x)$ introduced in Sect. 2. We define the subset $\mathcal{P}_T^{*f} \subseteq \mathcal{P}^{*f}$ of all testable physical propositions by setting

$$\mathcal{P}_T^{*f} = \{p_\alpha^f \mid \alpha(x) \in \Phi_T^*(x)\}.$$

Then, one gets that $\mathcal{P}_T^{*f}$ coincides with the set of all physical propositions associated with elementary wffs of $\Phi^*(x)$. Moreover, the posets $(\Phi_T^*(x)/\approx, \prec)$ (or $(\Psi_T^*/\equiv, <)$) and $(\mathcal{P}_T^{*f}, \subseteq)$ are order-isomorphic.

4 The Language of Properties $\mathcal{L}(x)$

Both in CM and in QM the set of all effects contains a subset of *decision effects* (see, e.g., [21], Chap. III) that we briefly call *properties* in this paper. Hence the set $\mathcal{E}^*$ of all predicates of $\mathcal{L}^*(x)$ contains a subset $\mathcal{E}$ of predicates denoting properties. Therefore one can consider the sublanguage $\mathcal{L}(x)$ of $\mathcal{L}^*(x)$ constructed by using only predicates in $\mathcal{E}$ and following the

[3] This isomorphism suggests that one could introduce the notion of *true with certainty* by firstly assigning $(\mathcal{P}^{*f}, \subseteq)$ with its algebraic structure and then connecting it with $\Phi^*(x)$, thus providing an *algebraic semantics* which allows one to avoid the introduction of a classical truth theory. One would thus follow standard procedures in QL, yet losing the links between two different notions of truth illustrated in this paper.

procedures summarized in Sect. 2. Thus, the set of all wffs of $\mathcal{L}(x)$, the set of all elementary wffs of $\mathcal{L}(x)$, the semantics and the physical interpretation of $\mathcal{L}(x)$, the logical preorder and equivalence on $\mathcal{L}(x)$, etc., are defined as in Sect. 2, simply dropping the suffix *. Hence one obtains that the poset $(\Phi(x)/\equiv, <)$ is a Boolean lattice and that the posets $(\Phi(x)/\approx, \prec)$ and $(\Psi/\equiv, <)$ are order-isomorphic. Moreover, the set $\Phi_T(x)$ of all testable wffs of $\mathcal{L}(x)$ is defined as follows,

$$\Phi_T(x) = \{\alpha(x) \in \Phi(x) \mid \exists E_\alpha \in \mathcal{E} : \alpha(x) \equiv E_\alpha(x)\},$$

and the posets $(\Phi_T(x)/\approx, \prec)$ and $(\Psi_T/\equiv, <)$ are order-isomorphic. It must be noted, however, that the notion of testability introduced in this way on $\Phi(x)$ does not coincide with the notion of testability following from the general definition in Sect. 2. Indeed, according to the latter, the set of all testable wffs of $\Phi(x)$ would be given by

$$\Phi'_T(x) = \{\alpha(x) \in \Phi(x) \mid \exists E_\alpha \in \mathcal{E}^* : \alpha(x) \equiv E_\alpha(x)\},$$

which implies $\Phi_T(x) \subseteq \Phi'_T(x)$, so that $\Phi_T(x)$ and $\Phi'_T(x)$ cannot, in general, be identified. Therefore we call *p-testability* the more restrictive notion of testability introduced here. We notice that the broadening of the set of testable wffs of $\Phi(x)$ following from considering the language of effects illustrates from our present viewpoint one of the known advantages of unsharp QM with respect to standard QM. Exploring this topic goes, however, beyond the scopes of the present paper.

Let us come now to propositions. The set $\mathcal{P}^f$ of all physical propositions associated with wffs of $\Phi(x)$ can be defined as in Sect. 3, replacing $\Phi^*(x)$ by $\Phi(x)$. Again, no change is required, but dropping the suffix *. Hence, proceeding as in Sect. 3, one can show that the posets $(\Phi(x)/\approx, \prec)$ (or $(\Psi/\equiv, <)$) and $(\mathcal{P}^f, \subseteq)$ are order-isomorphic. One can then introduce the subset

$$\mathcal{P}^f_T = \{p^f_\alpha \in \mathcal{P}^f \mid \alpha(x) \in \Phi_T(x)\} \subseteq \mathcal{P}^f$$

of all p-testable physical propositions and the subset

$$\mathcal{P}^{f\prime}_T = \{p^f_\alpha \in \mathcal{P}^f \mid \alpha(x) \in \Phi'_T(x)\} \subseteq \mathcal{P}^f$$

of all testable physical propositions (with $\mathcal{P}^f_T \subseteq \mathcal{P}^{f\prime}_T$). The distinction between $\mathcal{P}^f_T$ and $\mathcal{P}^{f\prime}_T$ is relevant in principle. However, we are only concerned with the subset $\mathcal{P}^f_T$ in the following. One easily gets, proceeding as in Sect. 3, that $\mathcal{P}^f_T$ coincides with the set of all physical propositions associated with elementary wffs of $\Phi(x)$, and that the posets $(\Phi_T(x)/\approx, \prec)$ (or $(\Psi_T/\equiv, <)$) and $(\mathcal{P}^f_T, \subseteq)$ are order-isomorphic.

5 Physical Propositions in Classical Mechanics

One can consider specific physical theories within the general scheme worked out in Sects. 2–4 by inserting in it suitable assumptions suggested by the intended interpretation in Sect. 2. In the case of CM, this leads to the collapse of a number of notions, which explains why some relevant conceptual differences have been overlooked in classical physics. Let us discuss briefly this issue.

First of all, all physical objects in a given state S possess the same properties according to CM. This feature can be formalized by introducing the following assumption.

CMS. The set $\mathcal{E}$ of all properties is such that, for every $E \in \mathcal{E}$ and $S \in \mathcal{S}$, either $\text{ext}_S E = \mathcal{U}_S$ or $\text{ext}_S E = \emptyset$.

Let us consider the language $\mathcal{L}(x)$ in CM. Because of axiom CMS, the restriction of the assignment function σ_S^ρ to $\Phi(x)$ does not depend on ρ, hence for every state S the wff $\alpha(x)$ is true iff the physical sentence $(\forall x)\alpha(x)$ associated with it is true. Thus, the notions of *true* and *certainly true* coincide on $\Phi(x)$. Hence, the logical preorder and equivalence on $\Phi(x)$ can be identified with the physical preorder and equivalence, respectively, so that the posets $(\Phi(x)/\approx, \prec)$ and $(\Psi/\equiv, <)$ can be identified with the Boolean lattice $(\Phi(x)/\equiv, <)$. Furthermore, all these posets are order-isomorphic to $(\mathcal{P}^f, \subseteq)$, which therefore is a Boolean lattice.

Secondly, let us consider p-testability. It is well known that, in principle, CM assumes that all properties can be simultaneously tested. This suggests one to introduce a further assumption, as follows.

CMT. The set $\Phi_T(x)$ of all p-testable wffs of $\Phi(x)$ coincides with $\Phi(x)$.

The above assumption implies $\Psi_T = \Psi$ and $\mathcal{P}_T^f = \mathcal{P}^f$. Hence, $(\mathcal{P}_T^f, \subseteq)$ is a Boolean lattice, which explains the common statement that "the logic of a classical mechanical system is a classical propositional logic" ([25], Chap. 5). However, this statement can be misleading, since it ignores a number of conceptual distinctions that we have pointed out in our general framework.

6 Physical Propositions in Quantum Mechanics

Assumption CMS does not hold in (standard, Hilbert space) QM. Indeed, if E denotes a property and S a state of the physical system Σ, the probability of getting result 1 (or 0) when performing a registration by means of a device belonging to E on a sample of Σ may be different both from 0 and from 1 in QM, which implies (via intended physical interpretation) that $\emptyset \neq \text{ext}_S E \neq \mathcal{U}_S$. Hence, one cannot conclude, as in CM, that $(\mathcal{P}^f, \subseteq)$ is a Boolean lattice. Moreover, there are properties in QM that cannot be simultaneously tested. Thus, neither assumption CMT holds, and one cannot assert that the sets $\mathcal{P}_T^f$ and $\mathcal{P}^f$ coincide. In order to discuss the order structure of $(\mathcal{P}_T^f, \subseteq)$ in QM, let us firstly introduce the symbols and notions that will be used in the following.

$\mathcal{H}$: the *Hilbert space* on the complex field associated with Σ.

$(\mathcal{L}(\mathcal{H}), \subseteq)$ (briefly, $\mathcal{L}(\mathcal{H})$): the complete, orthomodular, atomic lattice (which also has the covering property; see, e.g., [3], Chap. 10) of all *closed subspaces* of $\mathcal{H}$.

$^\perp$, $\Cap$ and $\Cup$: the *orthocomplementation*, the *meet* and the *join*, respectively, defined on $\mathcal{L}(\mathcal{H})$.

$\mathcal{A}$: the set of all *atoms* (one-dimensional subspaces) of $\mathcal{L}(\mathcal{H})$.

φ: the bijective mapping $\mathcal{S} \to \mathcal{A}$ of all (pure) states on the atoms of $\mathcal{L}(\mathcal{H})$.

χ: the bijective mapping $\mathcal{E} \to \mathcal{L}(\mathcal{H})$ of all properties on the closed subspaces of $\mathcal{L}(\mathcal{H})$.

$\prec$: the order on $\mathcal{E}$ canonically induced, via χ, by the order defined on $\mathcal{L}(\mathcal{H})$.

$^\perp$: the orthocomplementation on $\mathcal{E}$ canonically induced, via χ, by the orthocomplementation defined on $\mathcal{L}(\mathcal{H})$.

The mapping χ is an order isomorphism of $(\mathcal{E}, \prec)$ onto $(\mathcal{L}(\mathcal{H}), \subseteq)$ that preserves the orthocomplementation, hence $(\mathcal{E}, \prec)$ also is a complete, orthomodular, atomic lattice. We call it the *lattice of properties* of Σ, and identify it with a (standard, sharp) QL. We then introduce a further mapping

$$\theta\colon\ E \in \mathcal{E} \to \mathcal{S}_E = \{S \in \mathcal{S} \mid \varphi(S) \subseteq \chi(E)\} \in \mathcal{P}(\mathcal{S})$$

(where $\mathcal{P}(\mathcal{S})$ denotes the power set of $\mathcal{S}$) that associates every property $E \in \mathcal{E}$ with the set of states that are represented by atoms included in the subspace $\chi(E)$. Let $\mathcal{L}(\mathcal{S})$ be the range of θ. It is easy to see that also $(\mathcal{L}(\mathcal{S}), \subseteq)$ is a lattice, isomorphic to $(\mathcal{E}, \prec)$ and $(\mathcal{L}(\mathcal{H}), \subseteq)$. We still denote by $^{\perp}$ the orthocomplementation on $(\mathcal{L}(\mathcal{S}), \subseteq)$ canonically induced, via θ, by the orthocomplementation $^{\perp}$ defined on $(\mathcal{E}, \prec)$, and call $(\mathcal{L}(\mathcal{S}), \subseteq)$ *the lattice of all* $^{\perp}$*-closed subsets of* $\mathcal{S}$ (for, if $\mathcal{S}_E \in \mathcal{L}(\mathcal{S})$, $(\mathcal{S}_E^{\perp})^{\perp} = \mathcal{S}_E$).

The interpretations of $(\mathcal{E}, \prec)$ and $(\mathcal{L}(\mathcal{H}), \subseteq)$ then suggest identifying $\mathcal{L}(\mathcal{S})$ with the subset of all p-testable propositions. This can be formalized by introducing the following assumption.

QMT. Let $\alpha(x) \in \Phi_T(x)$, and let $E_\alpha \in \mathcal{E}$ be such that $\alpha(x) \equiv E_\alpha(x)$. Then, the physical proposition p_α^f of $\alpha(x)$ coincides with $\mathcal{S}_{E_\alpha}$ in QM.

Assumption QMT has some relevant immediate consequences. In particular, it implies that the equivalence relations $\equiv$ and $\approx$ coincide on $\Phi_T(x)$.[4] Indeed, note firstly that the bijectivity of the mapping χ entails that two properties $E, F \in \mathcal{E}$ coincide iff they are represented by the same subspace of $\mathcal{L}(\mathcal{H})$, hence iff $\mathcal{S}_E = \mathcal{S}_F$. Secondly, consider the wffs $\alpha(x)$, $\beta(x) \in \Phi_T(x)$ and let $E_\alpha, E_\beta \in \mathcal{E}$ be such that $\alpha(x) \equiv E_\alpha(x)$ and $\beta(x) \equiv E_\beta(x)$. Then, the following sequence of coimplications holds because of assumption QMT,

$$\begin{aligned}
&\alpha(x) \approx \beta(x) \quad \text{iff} \quad E_\alpha(x) \approx E_\beta(x) \quad \text{iff} \quad p_\alpha^f = p_\beta^f \quad \text{iff} \quad \mathcal{S}_{E_\alpha} = \mathcal{S}_{E_\beta} \quad \text{iff} \\
&E_\alpha = E_\beta \quad \text{iff} \quad E_\alpha(x) \equiv E_\beta(x) \quad \text{iff} \quad \alpha(x) \equiv \beta(x),
\end{aligned}$$

which proves our statement.

More important for our aims in this paper, assumption QMT implies that the poset $(\mathcal{P}_T^f, \subseteq)$ of all p-testable physical propositions associated with wffs of $\Phi_T(x)$ (equivalently, with elementary wffs of $\Phi(x)$) can be identified in QM with the lattice $(\mathcal{L}(\mathcal{S}), \subseteq)$ of all $^{\perp}$-closed subsets of $\mathcal{S}$. Hence the posets $(\Phi_T(x)/\approx, \prec)$ and $(\mathcal{P}_T^f, \subseteq)$, on one side, and the lattices $(\mathcal{L}(\mathcal{S}), \subseteq)$, $(\mathcal{L}(\mathcal{H}), \subseteq)$ and $(\mathcal{E}, \prec)$, on the other side, are order-isomorphic, and the isomorphisms preserve the orthocomplementation (on $(\mathcal{L}(\mathcal{S}), \subseteq)$, $(\mathcal{L}(\mathcal{H}), \subseteq)$ and $(\mathcal{E}, \prec)$) or canonically induce it (on $(\Phi_T(x)/\approx, \prec)$ and $(\mathcal{P}_T^f, \subseteq)$). We therefore denote orthocomplementation, meet and join in all these lattices by the same symbols (that is, $^{\perp}$, $\Cap$ and $\Cup$, respectively). Then, one can easily show that, for every $\alpha(x), \beta(x) \in \Phi_T(x)$,

$$\begin{aligned}
&\mathcal{S} \setminus p_\alpha^f \supseteq (p_\alpha^f)^{\perp} \in \mathcal{P}_T^f, \\
&p_\alpha^f \cap p_\beta^f = p_\alpha^f \Cap p_\beta^f \in \mathcal{P}_T^f, \\
&p_\alpha^f \cup p_\beta^f \subseteq p_\alpha^f \Cup p_\beta^f \in \mathcal{P}_T^f.
\end{aligned}$$

We can now state our main result in this section. Indeed, *the isomorphisms above allow one to recover (standard, sharp) QL as a quotient algebra of wffs of* $\mathcal{L}(x)$, *identifying it with* $(\Phi_T(x)/\approx, \prec)$. We stress that this identification has required four nontrivial steps: $^{(i)}$selecting p-testable wffs inside $\Phi(x)$; $^{(ii)}$grouping p-testable wffs into classes of physical rather than logical equivalence; $^{(iii)}$adopting assumption QMT; $^{(iv)}$identifying $(\mathcal{L}(\mathcal{S}), \subseteq)$ and $(\mathcal{E}, \prec)$.

[4]The coincidence of $\equiv$ and $\approx$ suggests that also the logical preorder $<$ and the physical preorder $\prec$ may coincide on $\Phi_T(x)$ in QM. Indeed, this coincidence has been introduced as an assumption within the general formulation of the SR interpretation of QM (see [14]). However, we do not need this assumption in the present paper.

The above result shows how the non-Boolean lattice of QL can be obtained without giving up classical semantics, which was our minimal aim in this paper. However, we have already seen in the Introduction that it has a deeper meaning if one accepts the SR interpretation of QM. Yet, it must be noted that no direct correspondence can be established between the logical operations on $\Phi(x)$ and the lattice operations of QL. By comparing the relations established in Sect. 3 and the relations above, one gets indeed that, for every $\alpha(x)$, $\beta(x)$, $\gamma(x) \in \Phi_T(x)$,

$$\alpha(x) \equiv \neg\beta(x) \quad \text{implies} \quad p_\alpha^f \subseteq \mathcal{S} \setminus p_\beta^f \supseteq (p_\beta^f)^\perp,$$

$$\alpha(x) \equiv \beta(x) \wedge \gamma(x) \quad \text{implies} \quad p_\alpha^f = p_\beta^f \cap p_\gamma^f = p_\beta^f \Cap p_\gamma^f,$$

$$\alpha(x) \equiv \beta(x) \vee \gamma(x) \quad \text{implies} \quad p_\alpha^f \supseteq p_\beta^f \cup p_\gamma^f \subseteq p_\beta^f \Cup p_\gamma^f$$

(see also [17]).

7 The Quantum Language $\mathcal{L}_{TQ}(x)$

The set $\Phi_T(x)$ generally is not closed with respect to $\neg$, $\wedge$ and $\vee$, in the sense that negation, meet and join of testable wffs may be not testable. However, we can construct a language $\mathcal{L}_{TQ}(x)$ whose wffs are testable and whose connectives correspond to lattice operations of QL, as follows.

(i) Let us take $\Phi_T(x)$ (equivalently, the set $\mathcal{E}(x)$ of all elementary wffs of $\Phi(x)$) as set of elementary wffs, and introduce three new connectives $\neg_Q$, $\wedge_Q$ and $\vee_Q$ (*quantum negation*, *quantum meet* and *quantum join*, respectively) and standard formation rules for *quantum well formed formulas* (briefly, *qwffs*).

(ii) Let $\Phi_{TQ}(x)$ be the set of all qwffs and let us define an assignment function τ_S^ρ on $\Phi_{TQ}(x)$ based on the assignment function σ_S^ρ defined on $\Phi(x)$. To this end, let us consider the wffs $\alpha(x)$, $\beta(x) \in \Phi_T(x)$ and let E_α, $E_\beta \in \mathcal{E}$ be such that $\alpha(x) \equiv E_\alpha(x)$ and $\beta(x) \equiv E_\beta(x)$. Then, for every $\rho \in \mathcal{R}$ and $S \in \mathcal{S}$, we put

$$\tau_S^\rho(\alpha(x)) = \sigma_S^\rho(\alpha(x)),$$

$$\tau_S^\rho(\neg_Q\alpha(x)) = t \text{ (or } f) \quad \text{iff} \quad \sigma_S^\rho(E_\alpha^\perp(x)) = t \text{ (or } f),$$

$$\tau_S^\rho(\alpha(x) \wedge_Q \beta(x)) = t \text{ (or } f) \quad \text{iff} \quad \sigma_S^\rho((E_\alpha \Cap E_\beta)(x)) = t \text{ (or } f),$$

$$\tau_S^\rho(\alpha(x) \vee_Q \beta(x)) = t \text{ (or } f) \quad \text{iff} \quad \sigma_S^\rho((E_\alpha \Cup E_\beta)(x)) = t \text{ (or } f).$$

It is apparent that $\neg_Q\alpha(x)$, $\alpha(x) \wedge_Q \beta(x)$ and $\alpha(x) \vee_Q \beta(x)$ are logically equivalent to wffs of $\Phi_T(x)$. Therefore the above semantic rules can be applied recursively by considering $\alpha(x)$, $\beta(x) \in \Phi_{TQ}(x)$, which defines τ_S^ρ on $\Phi_{TQ}(x)$. Hence, the notions of logical preorder $<$ and logical equivalence $\equiv$ can be extended to $\Phi_{TQ}(x)$, and every qwff is logically equivalent to a wff of $\mathcal{E}(x)$ (hence of $\Phi_T(x)$).

(iii) Let us associate a physical sentence $(\forall x)\alpha(x)$ with every qwff $\alpha(x) \in \Phi_{TQ}(x)$. Hence the notions of *certainly true*, *physical preorder* $\prec$ and *physical equivalence* $\approx$ can be introduced on $\Phi_{TQ}(x)$. Furthermore $\equiv$ and $\approx$ coincide on $\Phi_{TQ}(x)$, since they coincide on $\Phi_T(x)$ (Sect. 6).

(iv) For every $\alpha(x) \in \Phi_{TQ}(x)$, let us define the physical proposition $p_\alpha^f = \{S \in \mathcal{S} \mid \alpha(x)$ is certainly true in $S\}$ of $\alpha(x)$. Then, the set of all physical propositions associated with qwffs of $\Phi_{TQ}(x)$ coincides with $\mathcal{P}_T^f$. Moreover, the semantic rules established above entail that, for every $\alpha(x), \beta(x), \gamma(x) \in \Phi_{TQ}(x)$,

$$\alpha(x) \equiv \neg_Q \beta(x) \quad \text{iff} \quad p_\alpha^f = (p_\beta^f)^\perp,$$

$$\alpha(x) \equiv \beta(x) \wedge_Q \gamma(x) \quad \text{iff} \quad p_\alpha^f = p_\beta^f \Cap p_\gamma^f,\text{[5]}$$

$$\alpha(x) \equiv \beta(x) \vee_Q \gamma(x) \quad \text{iff} \quad p_\alpha^f = p_\beta^f \Cup p_\gamma^f.$$

(The proof of these coimplications is straightforward if one preliminarily notices that, for every $E, F \in \mathcal{E}$ the physical propositions of $E^\perp(x)$, $(E \Cap F)(x)$ and $(E \Cup F)(x)$ are $(p_E^f)^\perp$, $p_E^f \Cap p_F^f$ and $p_E^f \Cup p_F^f$, respectively, because of the definitions of $^\perp$, $\Cap$ and $\Cup$ on $\mathcal{E}$ and assumption QMT.)

We have thus constructed a language $\mathcal{L}_{TQ}(x)$ whose connectives correspond to lattice operations on QL, as desired. It must be stressed, however, that the semantic rules for quantum connectives have an empirical character since they depend on the empirical relations on the set of all properties, and that these rules coexist with the semantic rules for classical connectives in our approach.

Finally, we note that, for every $\alpha(x), \beta(x) \in \Phi_{TQ}(x)$, the following logical equivalence can be proved,

$$\alpha(x) \vee_Q \beta(x) \equiv \neg_Q((\neg_Q \alpha(x)) \wedge_Q (\neg_Q \beta(x))),$$

and a *quantum implication* connective $\to_Q$ can be introduced such that

$$\alpha(x) \to_Q \beta(x) \equiv (\neg_Q \alpha(x)) \vee_Q (\alpha(x) \wedge_Q \beta(x)).$$

The formal structure of the above logical equivalences is well known in QL. The novelty here is that $\alpha(x)$ and $\beta(x)$ are sentences referring to individual samples of physical objects, while the wffs of standard QL represent propositions and do not bear this interpretation.

8 Quantum Truth

The notion of *true with certainty* is defined in Sect. 2 for all wffs of $\mathcal{L}^*(x)$. Yet, only testable wffs of $\mathcal{L}^*(x)$ can be associated with empirical procedures that allow one to check whether they are certainly true or not.

For the sake of simplicity, let us restrict here to the sublanguage $\mathcal{L}(x)$ of $\mathcal{L}^*(x)$ and to the subset $\Phi_T(x) \subseteq \Phi_T^*(x)$ of p-testable wffs (Sect. 4). Then, the notion of *certainly true* can be worked out in QM in order to define a notion of quantum truth (briefly, *Q-truth*) on $\Phi_T(x)$, as follows.

QT. Let $\alpha(x) \in \Phi_T(x)$ and $S \in \mathcal{S}$. We put

$$\alpha(x) \text{ is } \textit{Q-true} \text{ in } S \quad \text{iff} \quad S \in p_\alpha^f,$$

[5] Note that, if $\alpha(x), \beta(x) \in \Phi_T(x)$, the second implication at the end of Sect. 6 shows that the physical proposition of $\alpha(x) \wedge \beta(x)$ is identical to the physical proposition of $\alpha(x) \wedge_Q \beta(x)$, which implies $\alpha(x) \wedge \beta(x) \approx \alpha(x) \wedge_Q \beta(x)$. Yet, one cannot assert in this case that $\alpha(x) \wedge \beta(x) \equiv \alpha(x) \wedge_Q \beta(x)$, since $\alpha(x) \wedge \beta(x)$ does not necessarily belong to $\Phi_T(x)$. The difference between $\wedge$ and $\wedge_Q$ was overlooked in a recent paper [17], and we thank S. Sozzo for bringing such issue to our attention.

$$\alpha(x) \text{ is } \textit{Q-false} \text{ in } S \quad \text{iff} \quad S \in (p_\alpha^f)^\perp,$$

$$\alpha(x) \text{ has no Q-truth value in } S \quad \text{iff} \quad S \in \mathcal{S} \setminus p_\alpha^f \cup (p_\alpha^f)^\perp.$$

Bearing in mind our definitions and results in Sects. 3, 4 and 6, we get

$$\alpha(x) \text{ is Q-true in } S \quad \text{iff} \quad \alpha(x) \text{ is certainly true in } S \quad \text{iff} \quad (\forall x)\alpha(x) \text{ is true in } S \quad \text{iff}$$
$$E_\alpha(x) \text{ is certainly true in } S \quad \text{iff} \quad (\forall x)E_\alpha(x) \text{ is true in } S.$$

The notion of Q-false has not yet an interpretation at this stage. However, we get from its definition

$$\alpha(x) \text{ is Q-false in } S \quad \text{iff} \quad E_\alpha^\perp(x) \text{ is certainly true in } S \quad \text{iff} \quad (\forall x)E_\alpha^\perp(x) \text{ is true in } S.$$

Let us remind now that, for every $E \in \mathcal{E}$, the property denoted by $E^\perp$ is usually interpreted in the physical literature as the equivalence class of registering devices obtained by reversing the roles of the outcomes 1 and 0 in all registering devices in E (we stress that we are considering properties here, not generic effects). This suggests one to add the following assumption to our scheme.

QMN. Let $E \in \mathcal{E}$. Then, $E^\perp(x) \equiv \neg E(x)$.

Assumption QMN implies

$$\alpha(x) \text{ is } \textit{Q-false} \text{ in } S \quad \text{iff} \quad (\forall x)\neg E_\alpha(x) \text{ is true in } S \quad \text{iff} \quad (\forall x)\neg\alpha(x) \text{ is true in } S \quad \text{iff}$$
$$\neg\alpha(x) \text{ is certainly true in } S,$$

hence we say that $\alpha(x)$ is *certainly false* in S iff it is Q-false in S.

The above terminology implies that $\alpha(x)$ has no Q-truth value in S iff $\alpha(x)$ is neither certainly true nor certainly false in S. We also say in this case that $\alpha(x)$ is *Q-indeterminate* in S.

It is now apparent that the notions of truth and Q-truth coexist in our approach. This realizes an *integrated perspective*, according to which the classical and the quantum notions of truth are not incompatible. Our approach also explains the "metaphysical disaster" mentioned in the Introduction [24] as following from attributing truth values that refer to quantified wffs of a first order predicate calculus to open wffs of the calculus itself.

Let us conclude our paper with some additional remarks.

Firstly, the notion of Q-truth introduced above applies to a fragment only (the set $\Phi_T(x) \subseteq \Phi(x)$) of the language $\mathcal{L}(x)$. If one wants to introduce this notion on the set of all wffs of a suitable quantum language, one can refer to the language $\mathcal{L}_{TQ}(x)$ constructed in Sect. 7. Then, all qwffs are testable, and definition QT can be applied in order to define Q-truth on $\mathcal{L}_{TQ}(x)$ by simply substituting $\Phi_{TQ}(x)$ to $\Phi_T(x)$ in it. Again, classical truth and Q-truth can coexist on $\mathcal{L}_{TQ}$ in our approach.

Secondly, definition QT can be physically justified by observing that most manuals and books on the foundations of QM introduce (usually implicitly) a verificationist notion of truth that can be summarized in our present terms as follows.

QVT. Let $\alpha(x) \in \Phi(x)$ and $S \in \mathcal{S}$. Then, $\alpha(x)$ is true (false) in S iff:

(i) $\alpha(x)$ is testable;

(ii) $\alpha(x)$ can be tested and found to be true (false) on a physical object in the state S without altering S.

It can be proved that the notion of truth introduced by definition QVT and the notion of Q-truth introduced by definition QT coincide. The proof is rather simple but requires some use of the theoretical apparatus of QM [16].

Finally, a further justification of definition QT can be given by noting that the notion of *true with certainty* translates in our context the notion of *certain*, or *true,* introduced in some partially axiomatized approaches to QM (as [23]).

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